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CYLINDRICAL SHELLS SUBJECT TO UNIFORM BENDING MOMENT AROUND AN ELLIPTIC HOLE

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Abstract—A theoretical analysis is presented for the membrane and bending stresses around an elliptic hole in an infinitely long, thin circular cylindrical shell. The major axis of the elliptic hole is taken to be perpendicular to shell axis. The shell is loaded by a uniformly distributed bending moment around the elliptic cutout. The method of solution involves perturbation in a curvature parameter. The results obtained are valid if the hole is small in size compared to the radius of the cylindrical shell. Expressions for the stresses at the tip of a circumferential and axial crack are also presented.

NOTATION

$2a$, $2b$	lengths of major and minor axes, respectively
$ce_n(\eta, q)$, $se_n(\eta, q)$	periodic Mathieu functions of cosine type and sine type, respectively
D	bending stiffness of shell wall, $\dfrac{Et^3}{12(1-\nu^2)}$
E	Young's modulus
F	complex stress–displacement function, $(W - im\Phi)$ where $i = \sqrt{-1}$
$2h$	distance between the foci
$H_n^{(1)}(v_1)$, $H_n^{(1)}(v_2)$	Hankel functions of first kind
$J_n(v_1)$, $J_n(v_2)$	Bessel functions of the first kind
K	scale factor for the elliptic coordinates $K = [h^2(\cosh 2\xi - \cos 2\eta)/2]^{\frac{1}{2}}$
M	uniformly distributed bending moment applied to the edge of the cutout
m	a constant, $\dfrac{[12(1-\nu^2)]^{\frac{1}{2}}}{Et^2}$
M_ξ, M_η, $M_{\xi\eta}$	bending moments in elliptic coordinates
$Me_n^{(1)}(\xi, q)$, $Ne_n^{(1)}(\xi, q)$	modified Mathieu functions
N_ξ, N_η, $N_{\xi\eta}$	membrane forces in elliptic coordinates
Q_ξ, Q_η	transverse shear forces in elliptic coordinates
R, t	radius and thickness of shell, respectively
r, θ	polar coordinates with center of the hole as origin, r being nondimensionalized through length of the semi-major axis, a
W	displacement normal to the middle surface of the shell, positive radially outwards
x, y	rectangular coordinates, nondimensionalized through length of the semi-major axis, a
α, ρ	polar coordinates with end of major axis as origin, ρ being nondimensionalized through length of the semi-major axis, a
β	a dimensionless curvature parameter, $\beta^2 = \dfrac{a^2[12(1-\nu^2)]^{\frac{1}{2}}}{8Rt}$
γ	Euler's constant, 0·5772 ...
λ	$(a-b)/(a+b)$

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$\sigma_x, \sigma_y, \tau_{xy}$	stresses in rectangular coordinates
ν	Poisson's ratio
Φ	stress function

1. INTRODUCTION

CIRCULAR cylindrical shells having lateral circular or non-circular openings, or intersected by branching shells are of common occurrence in a large number of engineering installations such as the fuselage of an aircraft, missile casings, boilers, reactors, deep-diving vehicles, pressure vessels and interstate oil lines. Although in recent years intensive studies have been made in this field [1–19] the present status of the area is in major deficiency and a great deal of theoretical and experimental work is still needed to develop methods of optimum design for unreinforced and reinforced openings.

Recently Murthy [20] obtained, by a power series expansion technique, the solution for the membrane and bending stresses around an elliptic hole in an infinitely long cylindrical shell subjected to axial tension. At present, more complete work still remains to be done regarding holes of various noncircular shapes. An elliptic hole is an important one because although circular holes are used widely in practice, an elliptic hole may have an advantage over the circular one in that it reduces the stress concentration. In an earlier paper Ariman and Rao [21] presented analytical solutions for the membrane and bending stresses around an elliptic hole in a circular cylindrical shell. The major axis of the ellipse was taken to be parallel to the axis of the cylinder. The shell was loaded by a uniformly distributed bending moment around the elliptic cutout.

In this paper, solutions are obtained for the stresses around an elliptic hole in a long, thin, circular cylindrical shell. The major axis of the ellipse is perpendicular to the shell axis. The shell is again loaded by a uniformly distributed bending moment around the edge of the elliptic cutout. Expressions for the stresses at the tip of a circumferential and axial crack are also presented. The method of solution which is similar to that described in [20, 21] involves a perturbation in a curvature parameter β given by

$$\beta^2 = a^2[12(1-\nu^2)]^{1/2}/8Rt$$

where $2a$ is the length of major axis of the ellipse, ν represents the Poisson's ratio and R and t are the radius and the thickness of the cylindrical shell, respectively. This quantity β which is treated as a dimensionless curvature parameter defines the size of the hole with respect to the dimensions of the shell. The hole is taken to be small enough so that $\beta \ll 1$. The solutions are expanded in series in even powers of β and products of $\ln \beta$ and even powers of β. These expansions are carried up to terms involving β^2 and $\beta^2 \ln \beta$, and terms involving β^4 and higher powers of β are neglected in comparison with unity. The solutions obtained in this paper give both the membrane and bending stresses and are valid for all values of eccentricity of the elliptic hole.

2. FORMULATION OF THE PROBLEM

For a thin circular cylindrical shell the governing complex partial differential equation of the Donnell theory is given as

$$\nabla^4 F + 8i\beta^2 \frac{\partial^2 F}{\partial x^2} = 0$$

where

$$F = W - im\Phi, \quad i = \sqrt{-1} \tag{2.1}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$
$$m = [12(1-\nu^2)]^{\frac{1}{2}}/Et^2. \tag{2.2}$$

Here F is the complex displacement function related to the radial displacement $W(x, y)$ and the stress function Φ and E is the Young's modulus. $W(x, y)$, displacement normal to the middle surface of the shell, is assumed to be positive radially outward.

In elliptic coordinates (Fig. 1) we have

$$x = \frac{h}{a}\sinh\xi\sin\eta, \qquad y = \frac{h}{a}\cosh\xi\cos\eta \tag{2.3}$$

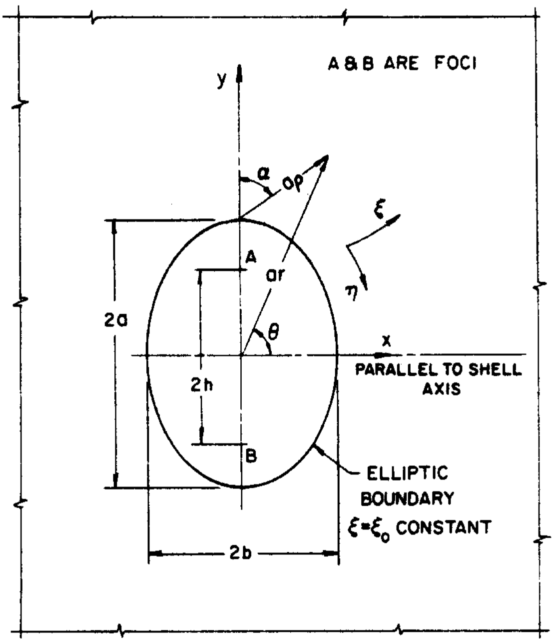

FIG. 1. Coordinate system.

and the expressions for stress resultants, stress couples and the transverse shears in terms of F in a general orthogonal curvilinear coordinate system can be written as

$$N_\xi = \frac{1}{K^4}\left[K^2\frac{\partial^2\Phi}{\partial\eta^2} + \frac{h^2}{2}\sinh 2\xi\frac{\partial\Phi}{\partial\xi} - \frac{h^2}{2}\sin 2\eta\frac{\partial\Phi}{\partial\eta}\right]$$

$$N_\eta = \frac{1}{K^4}\left[K^2\frac{\partial^2\Phi}{\partial\xi^2} - \frac{h^2}{2}\sinh 2\xi\frac{\partial\Phi}{\partial\xi} + \frac{h^2}{2}\sin 2\eta\frac{\partial\Phi}{\partial\eta}\right]$$

$$N_{\xi\eta} = \frac{1}{K^4}\left[-K^2\frac{\partial^2\Phi}{\partial\xi\,\partial n} + \frac{h^2}{2}\sin 2\eta\frac{\partial\Phi}{\partial\xi} + \frac{h^2}{2}\sinh 2\xi\frac{\partial\Phi}{\partial\eta}\right]$$

$$M_\xi = \frac{D}{K^4}\left[K^2\left(\frac{\partial^2 W}{\partial\xi^2} + \nu\frac{\partial^2 W}{\partial\eta^2}\right) + (1-\nu)\frac{h^2}{2}\sin 2\eta\frac{\partial W}{\partial\eta} - (1-\nu)\frac{h^2}{2}\sinh 2\xi\frac{\partial W}{\partial\xi}\right]$$

$$M_\eta = \frac{D}{K^4}\left[K^2\left(\frac{\partial^2 W}{\partial\eta^2} + \nu\frac{\partial^2 W}{\partial\xi^2}\right) - (1-\nu)\frac{h^2}{2}\sin 2\eta\frac{\partial W}{\partial\eta} + (1-\nu)\frac{h^2}{2}\sinh 2\xi\frac{\partial W}{\partial\xi}\right]$$

$$M_{\xi\eta} = \frac{D(1-\nu)}{K^4}\left[K^2\frac{\partial^2 W}{\partial\xi\,\partial\eta} - \frac{h^2}{2}\sinh 2\xi\frac{\partial W}{\partial\eta} - \frac{h^2}{2}\sin 2\eta\frac{\partial W}{\partial\xi}\right]$$

$$Q_\xi = \frac{D}{Ka^2}\frac{\partial}{\partial\xi}(\nabla^2 W)$$

$$Q_\eta = \frac{D}{Ka^2}\frac{\partial}{\partial\eta}(\nabla^2 W) \tag{2.4}$$

where

$$K = [h^2(\cosh 2\xi - \cos 2\eta)/2]^{\frac{1}{2}},$$

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{2.5}$$

and N_ξ, N_η, $N_{\xi\eta}$ are membrane forces in elliptic coordinates and M_ξ, M_η, $M_{\xi\eta}$; Q_ξ, Q_η represent stress couples and the transverse shears, respectively. The positive directions of these quantities are shown in (Fig. 2).

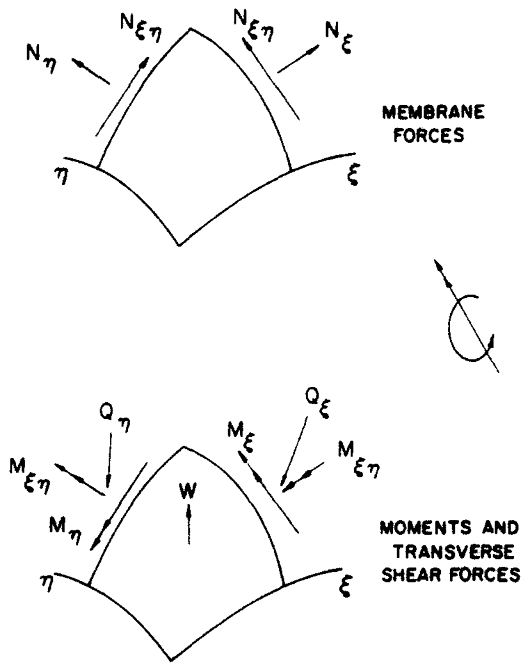

Fig. 2. Stress resultants.

The boundary conditions for an infinitely long cylindrical shell having, on its lateral surface, an elliptic hole subjected to edge loads require that we prescribe $N_\xi, N_{\xi\eta}, M_\xi$ and Q_ξ^* (Kirchhoff shear) along the edge of the hole $\xi = \xi_0$. In addition, at the two ends of the shell we must prescribe $N_\xi, N_{\xi\eta}, M_\xi, Q_\xi^*$. In this problem the only load is the uniform edge moment M, per unit length of the edge of the hole, so that

$$\begin{aligned} N_\xi &= 0 \\ N_{\xi\eta} &= 0 \\ M_\xi &= M \quad \text{at } \xi = \xi_0 \\ Q_\xi^* &= 0 \\ N_\xi &= N_{\xi\eta} = M_\xi = Q_\xi^* = 0, \quad \text{at } \xi = \infty \dagger \end{aligned} \tag{2.6}$$

where

$$Q_\xi^* = Q_\xi + \frac{1}{K}\frac{\partial M_{\xi\eta}}{\partial \eta}. \tag{2.7}$$

3. THE SOLUTION

The general solution of equation (2.1) appropriate to a residual problem in which the edge loads along the hole are symmetrically distributed with respect to the x and y axes can be expressed in a series form [21].

$$F = \sum_{n=0,1,2,\ldots}^{\infty}{}' (C_n + iD_n)(U_n + iV_n) \tag{3.1}$$

where

$$\begin{aligned} U_{2j} + iV_{2j} &= \cosh[(1-i)\beta x] Me_{2j}^{(1)}(\xi, q) ce_{2j}(\eta, q) \\ U_{2j+1} + iV_{2j+1} &= \sinh[(1-i)\beta x] Ne_{2j+1}^{(1)}(\xi, q) se_{2j+1}(\eta, q) \\ q &= i\beta^2 h^2/2a^2, \quad j = 0, 1, 2, \ldots. \end{aligned} \tag{3.2}$$

$Me_{2j}^{(1)}(\xi, q)$, $Ne_{2j+1}^{(1)}(\xi, q)$ are modified Mathieu functions and $ce_{2j}(\eta, q)$, $se_{2j+1}(\eta, q)$ are periodic Mathieu functions of cosine and sine types, respectively. C_n and D_n represent unknown real constants which must be determined from the first four of the set of boundary conditions. The conditions at $\xi = \infty$ are already taken care of with the selection of Mathieu functions $Me_{2j}^{(1)}$ and $Ne_{2j+1}^{(1)}$ which tend to zero as ξ approaches infinity.

It is the desired degree of accuracy that determines the number of terms to be considered in equation (3.1). In order to determine the stresses up to terms involving β^2 and $\beta^2 \ln \beta$, it is necessary to consider only the first five terms ($n = 0-4$). In expanding the solution, standard expansions for $ce_n(\eta, q)$ and $se_n(\eta, q)$ in powers of q are used [22]. The method of expansion used for $Me_{2j}^{(1)}(\xi, q)$ and $Ne_{2j+1}^{(1)}(\xi, q)$ is explained briefly in Appendix 1. In the neighborhood of the hole ($\xi = \xi_0$) U_n and V_n can be expanded for $n = 0$–4 in even powers of β and products of $\ln \beta$ and even powers of β. Due to the lengthiness of these expressions,

† $\xi = \infty$ is here to refer to points at a large enough distance from the hole that the edge effect is negligible.

they are not given here. Then it is found necessary to select the unknown coefficients C_n and D_n also in the form of power series in even powers of β and products of $\ln\beta$ and even powers of β, in the following form:

$$\begin{aligned}
C_0 &= \beta^2 \ln \beta C_0^{(1)} + \beta^2 C_0^{(2)} + \dots \\
D_0 &= D_0^{(0)} + \beta^2 D_0^{(2)} + \dots \\
C_1 &= C_1^{(0)} + \beta^2 C_1^{(2)} + \dots \\
D_1 &= \beta^2 \ln \beta D_1^{(1)} + \beta^2 D_1^{(2)} \dots \\
C_2 &= \beta^2 \ln \beta C_2^{(1)} + \beta^2 C_2^{(2)} + \dots \\
D_2 &= D_2^{(0)} + \beta^2 D_2^{(2)} + \dots \\
C_3 &= \beta^4 C_3^{(4)} + \dots \\
D_3 &= \beta^2 D_3^{(2)} + \dots \\
C_4 &= \beta^2 C_4^{(2)} + \dots \\
D_4 &= \beta^4 D_4^{(4)} + \dots .
\end{aligned} \tag{3.3}$$

Substituting equation (3.3) and the expressions for U_n and V_n into equation (3.1) and equating real and imaginary parts we obtain the expressions of W and Φ as

$$W = W_0 + \left(\frac{C_0^{(1)}}{2} - \frac{2}{\pi} D_0^{(2)} \right) \beta^2 \ln \beta + W_2 \beta^2 + 0(\beta^4) \tag{3.4}$$

$$\Phi = -\frac{D_0^{(0)}}{2m} + \Phi_1 \beta^2 \ln \beta + \Phi_2 \beta^2 + 0(\beta^4) \tag{3.5}$$

where

$$\begin{aligned}
W_0 &= W_0[D_0^{(0)}, C_1^{(0)}, D_2^{(0)}, \xi, \eta] \\
W_2 &= W_2[D_0^{(0)}, C_0^{(2)}, D_0^{(2)}, C_1^{(0)}, C_1^{(2)}, D_2^{(2)}, \xi, \eta] \\
\Phi_1 &= \Phi_1[D_0^{(0)}, C_0^{(1)}, C_0^{(2)}, C_1^{(0)}, D_1^{(1)}, C_2^{(1)}, \xi, \eta] \\
\Phi_2 &= \Phi_2[D_0^{(0)}, C_0^{(2)}, D_0^{(2)}, C_1^{(0)}, D_1^{(2)}, D_2^{(0)}, C_2^{(2)}, D_3^{(2)}, C_4^{(2)}, \xi, \eta].
\end{aligned} \tag{3.6}$$

As $\beta \to 0$, the radial displacement W given by equation (3.4) reduces to that of an infinitely long plate with an elliptic hole whose boundary is subjected to a uniformly distributed bending moment. Using the equations (3.4)–(3.6) and (2.4) the unknown coefficients can be evaluated from the first four of the boundary conditions given by equation (2.6). For brevity the expressions of the unknown coefficients are not presented here.

4. RESULTS AND DISCUSSION

The expressions for $N_\xi, N_\eta, N_{\xi\eta}, M_\xi, M_\eta, M_{\xi\eta}$ and Q_ξ, Q_η can now be obtained by substituting the expressions for W and Φ into equation (2.4). It is of particular interest to consider N_η, membrane force per unit length and M_η, stress couple per unit length, along the boundary of the hole ($\xi = \xi_0$)

$$[N_\eta]_{\xi=\xi_0} = \left\{\frac{48M\beta^2(1+\nu)}{t\sqrt{[12(1-\nu^2)]}(3+\nu)(\lambda+1)^2(1-2\lambda\cos 2\eta+\lambda^2)}\right\}$$
$$\{\psi_1^{(m)}(\beta,\lambda,\nu)+\psi_2^{(m)}(\beta,\lambda,\nu)\cos 2\eta+\psi_3^{(m)}(\lambda,\nu)\cos 4\eta\} \tag{4.1}$$

$$[M_\eta]_{\xi=\xi_0} = \left\{\frac{M}{(3+\nu)(1-2\lambda\cos 2\eta+\lambda^2)^2}\right\}\{\psi_1^{(b)}(\lambda,\nu)+\psi_2^{(b)}(\lambda,\nu)\cos 2\eta$$
$$+\psi_3^{(b)}(\lambda,\nu)\cos 4\eta\}+\left[\left\{\frac{\pi(1+\nu)M\beta^2}{(\lambda+1)^2(1-2\lambda\cos 2\eta+\lambda^2)^2(1-\nu)(3+\nu)^2}\right\}\right.$$
$$\left.\cdot\{\psi_4^{(b)}(\lambda,\nu)+\psi_5^{(b)}(\lambda,\nu)\cos 2\eta+\psi_6^{(b)}(\lambda,\nu)\cos 4\eta\}\right] \tag{4.2}$$

where

$$\psi_1^{(m)} = \left\{\gamma+\ln\left(\frac{\beta}{(\lambda+1)\sqrt{2}}\right)\right\}[(3+\nu)+4\lambda+(-2-2\nu)\lambda^2+(-2+2\nu)\lambda^3+(1-\nu)\lambda^4]$$
$$+[\tfrac{1}{2}(3+\nu)+\tfrac{1}{2}(3-\nu)\lambda+\tfrac{1}{6}(-29-7\nu)\lambda^2+\tfrac{5}{2}(-1+\nu)\lambda^3+\tfrac{3}{2}(1-\nu)\lambda^4+\tfrac{1}{6}(-1+\nu)\lambda^5]$$
$$\psi_2^{(m)} = \left\{\gamma+\ln\left(\frac{\beta}{(\lambda+1)\sqrt{2}}\right)\right\}[+(-3-\nu)+(-2+2\nu)\lambda+(1-\nu)\lambda^2]$$
$$+[+\tfrac{3}{4}(-3-\nu)+\tfrac{1}{6}(11+13\nu)\lambda+(4-\nu)\lambda^2+\tfrac{1}{2}(1-\nu)\lambda^3+\tfrac{1}{12}(-1+\nu)\lambda^4]$$
$$\psi_3^{(m)} = [\tfrac{1}{2}(-1-\nu)\lambda+\tfrac{1}{2}(-1+\nu)\lambda^2]$$
$$\psi_1^{(b)} = [(-3-\nu)+(-7+3\nu)\lambda^2+(1+3\nu)\lambda^4]$$
$$\psi_2^{(b)} = [8\lambda-8\nu\lambda^3]$$
$$\psi_3^{(b)} = [-2(1-\nu)\lambda^2]$$
$$\psi_4^{(b)} = [(9+6\nu+\nu^2)+(-1+2\nu+\nu^2)\lambda+(17+18\nu-3\nu^2)\lambda^2+(7-10\nu+3\nu^2)\lambda^3$$
$$+(-7+10\nu-3\nu^2)\lambda^4+(2-4\nu+2\nu^2)\lambda^5+(-1+2\nu-\nu^2)\lambda^5]$$
$$\psi_5^{(b)} = [(3-2\nu-\nu^2)+(-28-20\nu)\lambda+(-2+4\nu-2\nu^2)\lambda^2+(-2-20\nu+6\nu^2)\lambda^3$$
$$+(-9+14\nu-5\nu^2)\lambda^4+(6-8\nu+2\nu^2)\lambda^5] \tag{4.3}$$
$$\psi_6^{(b)} = [(-3+2\nu+\nu^2)\lambda+(10+8\nu-2\nu^2)\lambda^2+(5-6\nu+\nu^2)\lambda^3+(-4+4\nu)\lambda^4].$$

As $\beta \to 0$, equations (4.1) and (4.2) reduce to the flat plate solutions.

For the case of a circular hole, $\lambda = 0$ and $\eta = \pi/2-\theta$ where θ is the polar coordinate, (Fig. 1). Equations (4.1) and (4.2) then become

$$[N_\theta]_{r=a} = \frac{48M\beta^2(1+\nu)}{t\sqrt{[12(1-\nu^2)]}}[\{\tfrac{1}{2}+(\gamma+\ln[\beta/\sqrt{2})\}$$
$$+\cos 2\theta\{\tfrac{3}{4}+(\gamma+\ln[\beta/\sqrt{2}])\}] \tag{4.4}$$

$$[M_\theta]_{r=a} = -M+\frac{M\beta^2\pi(1+\nu)}{(1-\nu)(3+\nu)}[(3+\nu)+\cos 2\theta(-1+\nu)]. \tag{4.5}$$

Here, a, the length of the semi-major axis of the elliptic hole, becomes the radius of the circular hole in the limiting case. Equations (4.4) and (4.5) are identical to those obtained in Ref. [21]. In Figs. 3–5 membrane force N_η, perturbation part of the bending moment M_η and the total bending moment M_η are plotted as functions of the angular coordinate η and the geometrical parameters β and λ. The perturbation bending moment is defined as the difference between the bending moment in the shell and the moment given by the corresponding plate solution.

Figure 3 shows the membrane stress at the hole plotted as a function of λ and η for $\beta = 0{\cdot}2$ and $\nu = 0{\cdot}3$. In this figure the curve obtained by $\lambda = 0$, represents the circular

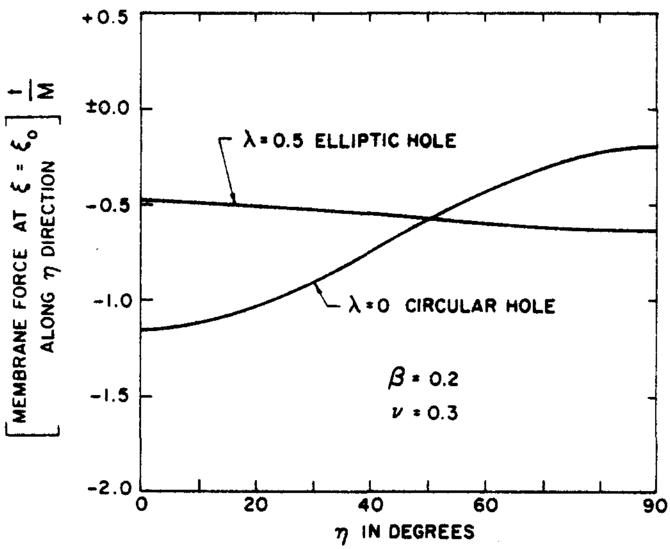

FIG. 3. Membrane force N_η at hole plotted as function of η and λ for $\beta = 0{\cdot}2$, $\nu = 0{\cdot}3$.

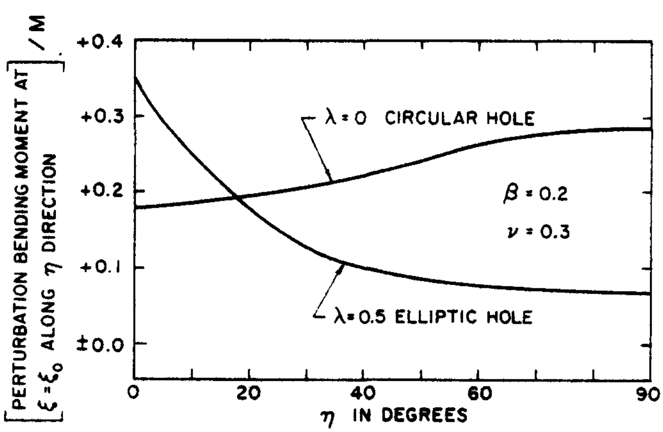

FIG. 4. Perturbation bending moment along η direction at hole plotted as function of η and λ for $\beta = 0{\cdot}2$, $\nu = 0{\cdot}3$.

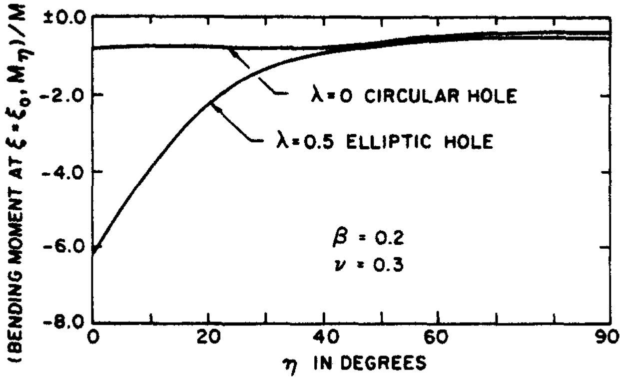

FIG. 5. Bending moment M_η at hole plotted as function of η and λ for $\beta = 0{\cdot}2$, $\nu = 0{\cdot}3$.

hole case. It is seen that for a larger eccentricity ($\lambda = 0{\cdot}5$), the membrane force N_η for the corresponding circular hole case becomes larger than that of an elliptic hole.

In Fig. 4 the variation of the perturbation bending moment M_η at the hole is plotted as a function of λ and η for $\beta = 0{\cdot}2$ and $\nu = 0{\cdot}3$. For the elliptic hole with $\lambda = 0{\cdot}5$, the perturbation bending moment tends to decrease with increasing values of the angular coordinate η. This is in contrast with the case of the corresponding circular hole. Approximately for $\eta > 17°$ the perturbation part of the bending moment M_η of the elliptic hole becomes smaller than that of the circular hole.

Figure 5 represents the variation of the total bending moment M_η at the hole as a function of λ and η for $\beta = 0{\cdot}2$ and $\nu = 0{\cdot}3$. It is seen that the eccentricity parameter λ has a considerable effect on the bending moment M_η. This effect becomes more significant particularly for $\eta < 40°$.

5. CIRCUMFERENTIAL AND AXIAL CRACKS

As a limiting case of the elliptic hole the circumferential and axial cracks may now be investigated without any major difficulty. Let ρ and α denote dimensionless polar coordinates with one of the ends of the major axis as the origin, Figs. 1 and 6. W and Φ which are known in elliptic coordinates ξ and η can be expressed in terms of ρ (ρ being nondimensionalized with respect to a) and α by using the expansions in Appendix 2, and after the substitution of $\xi_0 = 0$ for the case of a crack. In the vicinity of the crack tip where $\rho \ll 1$, the expressions for W and Φ are now obtained by a power series expansion in ρ. In these series, terms which are of the order of ρ^2 are neglected because they do not give rise to singular stresses.

1. Circumferential crack, the limiting case of an elliptic hole whose major axis is perpendicular to the shell axis. For this case W and Φ are given as:

$$W = \frac{3a^2(1+\nu)}{3+\nu}\frac{M}{Et^3}\left\{A \text{ const.} + A \text{ const.} (\rho\cos\alpha)\right.$$

$$\left. + \left[-4+\frac{\pi\beta^2}{2}\frac{(5+2\nu+\nu^2)}{(3+\nu)(1-\nu)}\right]\cdot\left(\frac{\rho^{3/2}}{\sqrt{2}}\right)\left[(1-\nu)\cos\frac{\alpha}{2}-\frac{7+\nu}{3}\cos\frac{3\alpha}{2}\right]+0(\rho^2)\right\} \tag{5.1}$$

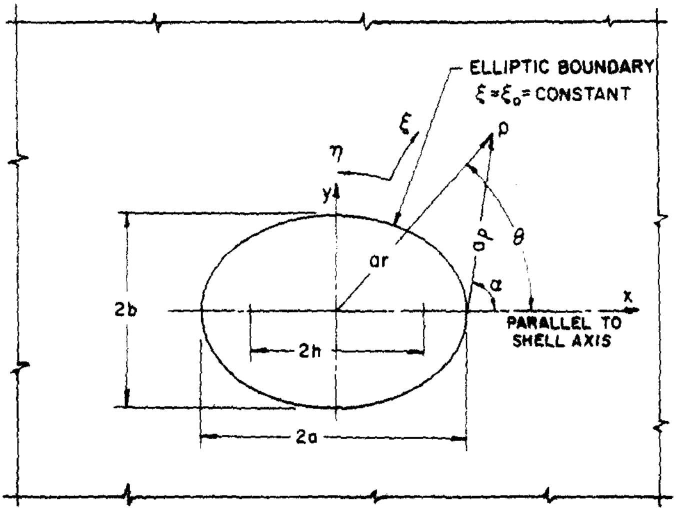

FIG. 6. Coordinate system.

$$\Phi = -\frac{1+\nu}{\sqrt{[12(1-\nu^2)](3+\nu)}}\frac{Ma^2}{t}\beta^2\Big\{A\text{ const.}+A\text{ const. }(\rho\cos\alpha)$$

$$+[(1+\nu)+2(1+\nu)(\gamma+\ln(\beta/2\sqrt{2}))]\cdot\left(\frac{\rho^{3/2}}{\sqrt{2}}\right)\left(3\cos\frac{\alpha}{2}+\cos\frac{3\alpha}{2}\right)+0(\rho^2)\Big\}. \qquad (5.2)$$

2. Axial crack, the limiting case of an elliptic hole whose major axis parallel to the shell axis.† For this case W and Φ are given in the following forms

$$W = \frac{3a^2(1+\nu)}{3+\nu}\frac{M}{Et^3}\Big[A\text{ const.}+A\text{ const. }(\rho\cos\alpha)$$

$$+\left\{-4+\frac{\pi\beta^2}{2}\cdot\frac{(1+2\nu+5\nu^2)}{(3+\nu)(1-\nu)}\right\}\cdot\left(\frac{\rho^{3/2}}{\sqrt{2}}\right)\left[(1-\nu)\cos\frac{\alpha}{2}-\frac{7+\nu}{3}\cos\frac{3\alpha}{2}\right]+0(\rho^2)\Big\} \qquad (5.3)$$

$$\Phi = -\frac{(1+\nu)}{\sqrt{[12(1-\nu^2)](3+\nu)}}\frac{Ma^2}{t}\beta^2\Big\{A\text{ const.}+A\text{ const. }(\rho\cos\alpha)$$

$$+\left[\frac{(5+37\nu)}{3}+2(1+5\nu)\left(\gamma+\ln(\beta/2\sqrt{2})\right)\right]\left(\frac{\rho^{3/2}}{\sqrt{2}}\right)\cdot\left(3\cos\frac{\alpha}{2}+\cos\frac{3\alpha}{2}\right)+0(\rho^2)\Big\}. \qquad (5.4)$$

In equations (5.1)–(5.4), the expressions of constants are not given since they do not contribute to stresses. Membrane forces and bending moments can now be written in

† The expressions of Φ and W are derived from those for the case of an elliptic hole whose major axis is parallel to the shell axis [21].

terms of Φ and W in orthogonal curvilinear coordinates ρ, α [23].

$$
\begin{aligned}
N_\rho &= \frac{1}{a^2}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\alpha^2}\right)\\
N_\alpha &= \frac{1}{a^2}\frac{\partial^2\Phi}{\partial\rho^2}\\
N_{\rho\alpha} &= -\frac{1}{a^2}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\alpha}\right)\\
M_\rho &= \frac{D}{a^2}\left[\frac{\partial^2 W}{\partial\rho^2}+\nu\left(\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\alpha^2}\right)\right]\\
M_\alpha &= \frac{D}{a^2}\left[\frac{1}{\rho}\frac{\partial W}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial\alpha^2}+\nu\frac{\partial^2 W}{\partial\rho^2}\right]\\
M_{\rho\alpha} &= \frac{(1-\nu)D}{a^2}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial W}{\partial\alpha}\right).
\end{aligned}
\tag{5.5}
$$

Then the membrane and bending components of stress resultants can be determined from equations (5.1)–(5.5). Since for crack problems it is usual to express the stresses in x, y coordinate system, we have the following membrane and bending stresses for the circumferential (case 1) and axial (case 2) cracks.

Membrane stresses:

case (1)
$$
\begin{aligned}
\sigma_x^{(m)} &= P_m^{(1)}\sigma_1^{(m)}\\
\sigma_y^{(m)} &= P_m^{(1)}\sigma_2^{(m)}\\
\tau_{xy}^{(m)} &= P_m^{(1)}\sigma_{12}^{(m)}
\end{aligned}
\tag{5.6}
$$

case (2)
$$
\begin{aligned}
\sigma_x^{(m)} &= P_m^{(2)}\sigma_2^{(m)}\\
\sigma_y^{(m)} &= P_m^{(2)}\sigma_1^{(m)}\\
\tau_{xy}^{(m)} &= P_m^{(2)}\sigma_{12}^{(m)}.
\end{aligned}
\tag{5.7}
$$

Bending stresses (extreme fibers):

case (1)
$$
\begin{aligned}
\sigma_x^{(b)} &= P_b^{(1)}\sigma_1^{(b)}\\
\sigma_y^{(b)} &= P_b^{(1)}\sigma_2^{(b)}\\
\tau_{xy}^{(b)} &= P_b^{(1)}\sigma_{12}^{(b)}
\end{aligned}
\tag{5.8}
$$

case (2)
$$
\begin{aligned}
\sigma_x^{(b)} &= P_b^{(2)}\sigma_2^{(b)}\\
\sigma_y^{(b)} &= P_b^{(2)}\sigma_1^{(b)}\\
\tau_{xy}^{(b)} &= P_b^{(2)}\sigma_{12}^{(b)}
\end{aligned}
\tag{5.9}
$$

where

$$\sigma_1^{(m)} = \frac{1}{\sqrt{(2\rho)}}\left[\frac{5}{4}\cos\frac{\alpha}{2} - \frac{1}{4}\cos\frac{5\alpha}{2}\right] + 0(\rho^0)$$

$$\sigma_2^{(m)} = \frac{1}{\sqrt{(2\rho)}}\left[\frac{3}{4}\cos\frac{\alpha}{2} + \frac{1}{4}\cos\frac{5\alpha}{2}\right] + 0(\rho^0) \quad (5.10)$$

$$\sigma_{12}^{(m)} = \frac{1}{\sqrt{(2\rho)}}\left[-\frac{1}{4}\sin\frac{\alpha}{2} + \frac{1}{4}\sin\frac{5\alpha}{2}\right] + 0(\rho^0)$$

$$\sigma_1^{(b)} = \pm\left(-\frac{1}{\sqrt{(2\rho)}}\right)\left[\frac{(11+5\nu)}{4}\cos\frac{\alpha}{2} + \frac{(1-\nu)}{4}\cos\frac{5\alpha}{2}\right] + 0(\rho^0)$$

$$\sigma_2^{(b)} = \pm\left(\frac{1}{\sqrt{(2\rho)}}\right)\left[\frac{(3-3\nu)}{4}\cos\frac{\alpha}{2} + \frac{(1-\nu)}{4}\cos\frac{5\alpha}{2}\right] + 0(\rho^0) \quad (5.11)$$

$$\sigma_{12}^{(b)} = \pm\left(-\frac{1}{\sqrt{(2\rho)}}\right)\left[-\frac{(7+\nu)}{4}\sin\frac{\alpha}{2} - \frac{(1-\nu)}{4}\sin\frac{5\alpha}{2}\right] + 0(\rho^0)$$

$$P_m^{(1)} = -\frac{\sqrt{[12(1-\nu^2)]}}{(3+\nu)(1-\nu)}\left(\frac{M}{t^2}\right)\left(\frac{\beta^2}{4}\right)[(1+\nu) + 2(1+\nu)(\gamma + \ln[\beta/2\sqrt{2}])] + 0(\beta^4) \quad (5.12)$$

$$P_m^{(2)} = -\frac{\sqrt{[12(1-\nu^2)]}}{(3+\nu)(1-\nu)}\cdot\left(\frac{M}{t^2}\right)\left(\frac{\beta^2}{4}\right)\left[\frac{(5+37\nu)}{3} + 2(1+5\nu)(\gamma + \ln[\beta/2\sqrt{2}])\right] + 0(\beta^4) \quad (5.13)$$

$$P_b^{(1)} = \left[1 - \frac{(5+2\nu+\nu^2)}{(3+\nu)(1-\nu)}\frac{\pi\beta^2}{8}\right]\left[\frac{6M}{t^2(3+\nu)}\right] + 0(\beta^4) \quad (5.14)$$

$$P_b^{(2)} = \left[1 - \frac{(1+2\nu+5\nu^2)}{(3+\nu)(1-\nu)}\frac{\pi\beta^2}{8}\right]\left[\frac{6M}{t^2(3+\nu)}\right] + 0(\beta^4). \quad (5.15)$$

The positive and negative signs at the beginning of equation (5.11) refer to inner and outer surfaces of the shell, respectively. Equations (5.6)–(5.15) are identical with those obtained by Folias† [24, 25]. The method of analysis used by Folias involves the solution of a set of coupled singular integral equations of Cauchy type and is therefore, different from the one used here.

There are certain interesting features about the stresses near the crack tip:

1. Both membrane and bending stresses at the crack tip exhibit the same kind of inverse square root singularity as found in plates.

2. Angular distributions of bending and membrane stresses near the crack tip are dependent on Poisson's ratio as in the case of plates. It will be recalled that the angular membrane stress distribution near the crack tip of a circumferential crack in a circular cylindrical shell subjected to axial tension does not depend on Poisson's ratio [26].

3. The expressions for bending stresses of case 2 can also be obtained from those for case 1 by merely replacing the quantity $(5+2\nu+\nu^2)$ appearing in the second term of equation (5.14) by $(1+2\nu+5\nu^2)$ and by interchanging $\sigma_1^{(b)}$ and $\sigma_2^{(b)}$ in equation (5.8).

† Taking a typographical error in Ref. [24] into account.

6. CONCLUDING REMARKS

In this paper stresses around an elliptic hole in a circular cylindrical shell are analyzed by a power series expansion technique. The major axis of the elliptic hole is taken to be perpendicular to shell axis. The shell is loaded by a uniformly distributed bending moment around the elliptic cutout. The method of solution involves perturbation in a curvature parameter β. The results obtained are valid if the hole is small in size compared to the radius of the cylindrical shell (approximately $\beta < 0{\cdot}3$). The solutions given in this paper are not restricted to nearly circular elliptic holes [27].

The solutions derived here in the limiting cases reduce to the well known solutions. As $\beta \rightarrow 0$, the solutions (4.1) and (4.2) reduce to the ones for an elliptic hole in a flat plate with the same bending moment loading. With $\lambda = 0$ equations (4.1) and (4.2) become the corresponding expression for the case of a circular hole in the same circular cylindrical shell. Finally, the expressions for membrane and bending stresses for circumferential and axial cracks are also presented.

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APPENDIX 1

If

$$V_1 = q^{\frac{1}{2}}\,e^{-\xi} \quad \text{and} \quad V_2 = q^{\frac{1}{2}}\,e^{\xi},$$

where

$$q = i\frac{\beta^2 h^2}{2a^2},$$

then,

$$\begin{aligned}
Me_{2j}^{(1)}(\xi, q) &= \sum_{K=0,1,2,\ldots}^{\infty} (-1)^K A_{2K}^{(2j)} J_K(V_1) H_K^{(1)}(V_2) \\
Ne_{2j+1}^{(1)}(\xi, q) &= \sum_{K=0,1,2,\ldots}^{\infty} (-1)^K B_{2K+1}^{(2j+1)} [J_K(V_1) H_{K+1}^{(1)}(V_2) - J_{K+1}(V_1) H_K^{(1)}(V_2)] \\
ce_{2j}(\eta, q) &= \sum_{K=0,1,2,\ldots}^{\infty} A_{2K}^{(2j)} \cos 2K\eta \\
se_{2j+1}(\eta, q) &= \sum_{K=0,1,2,\ldots}^{\infty} B_{2K+1}^{(2j+1)} \sin(2K+1)\eta.
\end{aligned} \tag{A.1}$$

Here J_K and $H_K^{(1)}$ represent Bessel and Hankel functions of the first kind, respectively. Standard expansions for the characteristic coefficients $A_{2K}^{(2j)}$, $B_{2K+1}^{(2j+1)}$ in powers of q are available in Ref. [22]. It should be noted that the formulas for Mathieu and modified Mathieu functions given in this Appendix are the same as those given by MacLachlan [22] except that they are not normalized. Normalization is not necessary because these functions are multiplied by arbitrary constants to be determined from boundary conditions.

APPENDIX 2

If $\rho \ll 1$, the following expansions are valid:

$$\cos 2\eta = 1 - 4\rho \sin^2\frac{\alpha}{2} + 0(\rho^2)$$

$$e^{-2\xi} = 1 - 2\sqrt{(2\rho)}\cos\frac{\alpha}{2} + 4\rho\cos^2\frac{\alpha}{2} - \frac{1}{\sqrt{2}}\rho^{\frac{3}{2}}\cos\frac{\alpha}{2}\left(1 + 4\cos^2\frac{\alpha}{2}\right) + 0(\rho^2)$$

$$e^{2\xi} = 1+2\sqrt{(2\rho)}\cos\frac{\alpha}{2}+4\rho\cos^2\frac{\alpha}{2}+\frac{1}{\sqrt{2}}\rho^{\frac{3}{2}}\cos\frac{\alpha}{2}\left(1+4\cos^2\frac{\alpha}{2}\right)+0(\rho^2)$$

$$\xi = \sqrt{(2\rho)}\cos\frac{\alpha}{2}-\frac{1}{6\sqrt{2}}\rho^{\frac{3}{2}}\cos\frac{3\alpha}{2}+0(\rho^2). \quad (A.2)$$

Expansions for other functions of ξ and η which are required can easily be derived from equation (A.2).

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Абстракт—Даётся теоретический анализ для напряжений в безмоментном состоянии и напряжений от изгиба, вокруг эллиптического отверстия, в бесконечно длинной, тонкой, круглой цилиндрической оболочке. Большая ось эллиптического отверстия перпендикулярна к оси оболочки. Оболочка нагружена равномерно распределённым моментом изгиба, вокруг эллипртического контура. Метод решения заключает возмущение в параметре кривизны. Полученные решения являются важными для случая малого ответстия, по сравнению с радиусом цилиндриуеской оюолочки. Даются также выражения для напряжений в вершине окружной и осевой щели.